

A Note on the Expected Length of the Longest Common Subsequences of two i.i.d. Random Permutations

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Abstract

The minimal expected length of the Longest Common Subsequences (*LCSs*) of two i.i.d. random permutations of the symmetric group is not attained in the uniform case.

1 Introduction

The length of the Longest Increasing Subsequences (*LISs*) of a uniform random permutation $\sigma \in \mathcal{S}_n$ is well studied and we refer to [4] for precise results and a comprehensive bibliography on this subject. Recently, [3] showed that for two independent random permutations $\sigma_1, \sigma_2 \in \mathcal{S}_n$, and as long as σ_1 is uniformly distributed and regardless of the distribution of σ_2 , the length of the *LCSs* of the two permutations is identical in law to the length of the *LISs* of σ_1 , i.e. $LCS(\sigma_1, \sigma_2) =^{\mathcal{L}} LIS(\sigma_1)$. It

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is therefore rather natural to study the behavior of $LCS(\sigma_1, \sigma_2)$, when σ_1 and σ_2 are i.i.d. but not necessarily uniform. This was already done by Bukh and Zhou in [2], who raised some questions which can be rephrased as follows:

Conjecture/Interrogation. *Let P be an arbitrary probability distribution on \mathcal{S}_n . Let σ_1 and σ_2 be two i.i.d. permutations sampled from P . Then $\mathbb{E}_P[LCS(\sigma_1, \sigma_2)] \geq \sqrt{n}$. It might even be true that the uniform distribution U on \mathcal{S}_n gives a minimizer.*

Below we prove the suboptimality of the uniform distribution by explicitly building a distribution having a smaller expectation. In the next section, before presenting and proving our main result, we give a few definitions and formalize this minimizing problem as a quadratic programming one. Section 3 further explore some properties of the spectrum of the coefficient matrix of our quadratic program. In the concluding section, a quick cubic root lower bound is given along with a few pointers for future research.

2 Main Results

We begin with a few notations. Throughout, σ and π are, respectively, used for random and deterministic permutations. By convention, $[n] := \{1, 2, 3, \dots, n\}$ and so $\{\pi_i\}_{i \in [n!]} = \mathcal{S}_n$ is a particular ordered enumeration of \mathcal{S}_n . (Some other orderings of \mathcal{S}_n will be given when necessary.) Next, a random permutation σ is said to be sampled from $P = (p_i)_{i \in [n!]}$, if $\mathbb{P}_P(\sigma = \pi_i) = p_i$. The uniform distribution is therefore $U = (1/n!)_{i \in [n!]}$ and, for simplification, it is denoted by $E/n!$, where $E = (1)_{i \in [n!]}$ is the n -tuple only made up of ones. When needed, a superscript will indicate the degree of the symmetric group we are studying, e.g., $\sigma^{(n)}$ and $P^{(n)}$ are respectively a random permutation and distribution from \mathcal{S}_n .

Let us now formalize the expectation as a quadratic form:

$$\begin{aligned} \mathbb{E}_P[LCS(\sigma_1, \sigma_2)] &= \sum_{i, j \in [n!]} p_i LCS(\pi_i, \pi_j) p_j \\ &= \sum_{i, j \in [n!]} p_i \ell_{ij} p_j \\ &= P^T L^{(n)} P, \end{aligned} \tag{2.1}$$

where $\ell_{ij} := LCS(\pi_i, \pi_j)$ and $L^{(n)} := \{\ell_{ij}\}_{(i,j) \in [n!] \times [n]}$. It is clear that $\ell_{ij} = \ell_{ji}$ and that $\ell_{ii} = n$. A quick analysis of the cases $n = 2$ or 3 shows that both $L^{(2)}$ and $L^{(3)}$ are positive semi-definite. However, this property does not hold further:

Lemma 2.1. *For $n \geq 4$, the smallest eigenvalue $\lambda_1^{(n)}$ of $L^{(n)}$ is negative.*

Proof. Linear algebra gives $\lambda_1^{(2)} = 1$ and $\lambda_1^{(3)} = 0$. So to prove the result, it suffices to show that $\lambda_1^{(k+1)} < \lambda_1^{(k)}$, $k \geq 1$ and this is done by induction. The base case is true, since $\lambda_1^{(2)} = 1 > 0 = \lambda_1^{(3)}$. To reveal the connection between $L^{(k+1)}$ and $L^{(k)}$, the enumeration of \mathcal{S}_{k+1} is iteratively built on that of \mathcal{S}_k by inserting the new element $(k+1)$ into the permutations from \mathcal{S}_k in the following way: the enumeration of the $(k+1)!$ permutations is split into $(k+1)$ trunks of equal size $k!$. In the i th trunk, the new element $(k+1)$ is inserted behind the $(k+1-i)$ th digit in the permutation from \mathcal{S}_k . For example, if \mathcal{S}_2 is enumerated as $\{[12], [21]\}$, then the first trunk of enumeration in \mathcal{S}_3 is $\{[123], [213]\}$, the second is $\{[132], [231]\}$ and the third is $\{[312], [321]\}$. Then the overall enumeration for \mathcal{S}_3 is $\{[123], [213], [132], [231], [312], [321]\}$.

By this enumeration, the principal minor of size $k! \times k!$ is row and column indexed by the enumeration of the permutations $\{\pi_i^{(k)}\}_{i \in [k!]}$ from \mathcal{S}_k with $(k+1)$ as the last digit, i.e., $\{[\pi_i^{(k)}(k+1)]\}_{i \in [k!]} \subseteq \mathcal{S}_{k+1}$. Then the (i, j) entry of the submatrix is

$$LCS([\pi_i(k+1)], [\pi_j(k+1)]) = LCS(\pi_i, \pi_j) + 1,$$

since the last digit $(k+1)$ adds an extra element into the longest common subsequences. Hence, the $k! \times k!$ principal minor of $L^{(k+1)}$ is $L^{(k)} + E^{(k)}(E^{(k)})^T$, where $E^{(k)}$ is the vector of $\mathbb{R}^{k!}$ only made up of ones. Moreover, $E^{(k)}$ is actually a right eigenvector of $L^{(k)}$. Indeed, it suffices to show that all the row sums of $L^{(k)}$ share the same value. But the sum of the π_i -indexed row of $L^{(k)}$ is

$$\begin{aligned} \sum_{j \in [k!]} LCS(\pi_i, \pi_j) &= \sum_{j \in [k!]} LCS(id, \pi_i^{-1} \pi_j) \\ &= \sum_{j \in [k!]} LIS(\pi_i^{-1} \pi_j), \end{aligned}$$

because relabeling π_i and π_j simultaneously does not change the length of the LCS s and because a particular relabeling to make π_i to be the identity permutation, which is equivalent to left composition by π_i^{-1} , is applied here. Further, any LCS of the

identity permutation and $\pi_i^{-1}\pi_j$ is a *LIS* of $\pi_i^{-1}\pi_j$ and vice versa. So the row sum is equal to

$$\sum_{j \in [k!]} LIS(\pi_i^{-1}\pi_j) = \sum_{\pi \in \mathcal{S}_k} LIS(\pi),$$

since left composition by π_i^{-1} is a bijection from \mathcal{S}_k to \mathcal{S}_k . This row sum is also the eigenvalue associated with $E^{(k)}$, which is also distinct from the smallest eigenvalue $\lambda_1^{(k)} \leq 0$.

On the other hand, since $L^{(k)}$ is symmetric, the eigenvectors $R_1^{(k)}$ and $E^{(k)}$ associated with the eigenvalues $\lambda_1^{(k)}$ and $\sum_{\pi \in \mathcal{S}_k} LIS(\pi)$ are orthogonal, i.e.,

$$(E^{(k)})^T R_1^{(k)} = 0. \quad (2.2)$$

Without loss of generality, let $R_1^{(k)}$ be a unit vector, then from (2.2),

$$\begin{aligned} \lambda_1^{(k)} &= (R_1^{(k)})^T L^{(k)} (R_1^{(k)}) \\ g &= (R_1^{(k)})^T (L^{(k)} + E^{(k)}(E^{(k)})^T) R_1^{(k)}. \end{aligned} \quad (2.3)$$

As $L^{(k)} + E^{(k)}(E^{(k)})^T$ is the $k! \times k!$ principal minor of $L^{(k+1)}$, (2.3) becomes

$$\begin{bmatrix} R_1^{(k)} \\ 0 \end{bmatrix}^T L^{(k+1)} \begin{bmatrix} R_1^{(k)} \\ 0 \end{bmatrix} \geq \min_{R^T E=0, \|R\|=1} R^T L^{(k+1)} R = \lambda_1^{(k+1)}, \quad (2.4)$$

where $R_1^{(k)}$ is properly extended to $\begin{bmatrix} R_1^{(k)} \\ 0 \end{bmatrix} \in \mathbb{R}^{(k+1)!}$ and where the above inequality

holds true since $\begin{bmatrix} R_1^{(k)} \\ 0 \end{bmatrix}^T E^{(k+1)} = (R_1^{(k)})^T E^{(k)} = 0$ and $\left\| \begin{bmatrix} R_1^{(k)} \\ 0 \end{bmatrix} \right\| = \|R_1^{(k)}\| = 1$,

where $\|\cdot\|$ is the Euclidean norm. Moreover, equality in (2.4) holds if and only if

$\begin{bmatrix} R_1^{(k)} \\ 0 \end{bmatrix}$ is a eigenvector of $L^{(k+1)}$ associated with $\lambda_1^{(k+1)}$.

Next, we show, by contradiction, that this cannot be the case. Indeed, assume that

$$L^{(k+1)} \begin{bmatrix} R_1^{(k)} \\ 0 \end{bmatrix} = \lambda_1^{(k+1)} \begin{bmatrix} R_1^{(k)} \\ 0 \end{bmatrix}. \quad (2.5)$$

Now, the $k! \times k!$ submatrix at the bottom-left corner of $L^{(k+1)}$, which is row-indexed by $\{[(k+1)\pi_i]\}_{i \in [k!]}$ and column-indexed by $\{[\pi_i(k+1)]\}_{i \in [k!]}$, is actually equivalent

to $L^{(k)}$. Indeed, the (i, j) -entry of this submatrix is

$$LCS([(k+1)\pi_i], [\pi_j(k+1)]) = LCS(\pi_i, \pi_j),$$

since $(k+1)$ can be in some LCS only if the length of this LCS is 1. The vector made of the bottom $k!$ elements on the left-hand-side of (2.5) is $L^{(k)}R_1^{(k)} = \lambda_1^{(k)}R_1^{(k)}$, which is a non-zero vector. However, on the right-hand-side, the corresponding bottom $k!$

elements of the vector $\begin{bmatrix} R_1^{(k)} \\ 0 \end{bmatrix}$ form the zero vector. This leads to a contradiction.

So,

$$\lambda_1^{(2)} = 1 > 0 = \lambda_1^{(3)} > \lambda_1^{(4)} > \lambda_1^{(5)} \dots$$

□

This smallest negative eigenvalue and its associated eigenvector will help build a distribution on \mathcal{S}_n , for which the LCS s have a smaller expectation than for the uniform one.

Theorem 2.2. *Let σ_1 and σ_2 be two i.i.d. random permutations sampled from a distribution P on the symmetric group \mathcal{S}_n . Then, for $n \leq 3$, the uniform distribution U minimizes $\mathbb{E}_p[LCS(\sigma_1, \sigma_2)]$, while, for $n \geq 4$, U is sub-optimal.*

Proof. As we have seen in (2.1),

$$\begin{aligned} \mathbb{E}_P[LCS(\sigma_1, \sigma_2)] &= P^T L P \\ &= (P - U)^T L (P - U) + 2P^T L U - U^T L U \\ &= (P - U)^T L (P - U) + 2U^T L U - U^T L U \\ &= (P - U)^T L (P - U) + U^T L U, \end{aligned} \tag{2.6}$$

where $P^T L U = U^T L U$, since U is an eigenvector of L and $P^T U = 1$.

When $n = 2, 3$, $L^{(n)}$ is positive semi-definite and therefore $(P - U)^T L (P - U) \geq 0$. So, $P^T L P \geq U^T L U$.

However, when $n \geq 4$, by Lemma 2.1, the smallest eigenvalue $\lambda_1^{(n)}$ is strictly negative and the associated eigenvector $R_1^{(n)}$ is such that $U^T R_1^{(n)} = 0 = E^T R_1^{(n)}$. Hence, there exists a positive constant c such that $cR_1^{(n)} \succeq -1/n!$, where \succeq stands for componentwise inequality. Let P_0 be such that $P_0 - U = cR_1^{(n)}$. It immediately follows that

$$E^T P_0 = E^T (U + cR_1^{(n)}) = 1 + 0 = 1,$$

and also

$$P_0 = U + cR_1^{(n)} \succeq 0.$$

Therefore, P_0 is a well-defined distribution on \mathcal{S}_n . On the other hand, by (2.6), the expectation under P_0 is

$$\begin{aligned} \mathbb{E}_{P_0}[LCS(\sigma_1, \sigma_2)] &= (P_0 - U)^T L(P_0 - U) + U^T L U \\ &= c^2 (R_1^{(n)})^T L(R_1^{(n)}) + U^T L U \\ &= c^2 \lambda_1^{(n)} + U^T L U \\ &< U^T L U, \end{aligned}$$

but the above right-hand side is the expectation under the uniform distribution, namely, $\mathbb{E}_U[LCS(\sigma_1, \sigma_2)]$. \square

The existence of negative eigenvalues contributes to the above construction and to the corresponding counterexample. So, as a next step, properties of this smallest negative eigenvalue and of the spectrum of the coefficient matrix $L^{(n)}$ are explored.

3 Further Properties of L

As we have seen, the vector $E^{(n)}$ which is made up of only ones is an eigenvector associated with the eigenvalue $\sum_{\pi \in S_n} LIS(\pi)$. It is not hard to show that this eigenvalue is, in fact, the radius of the spectrum of $L^{(n)}$.

Proposition 3.1. $\sum_{\pi \in S_k} LIS(\pi)$ is the spectral radius of $L^{(n)}$.

Proof. Without loss of generality, let (λ, R) be a pair of eigenvalue and eigenvector to $L^{(n)}$ such that $\max_{i \in [n!]} |r_i| = 1$, where $R = (r_1, \dots, r_{n!})^T$ and let i_0 be the index such that $|r_{i_0}| = 1$. Let us focus now on the i_0 th element of λR . Then, since $L^{(n)} R = \lambda R$,

$$\begin{aligned} |\lambda| &= |\lambda r_{i_0}| \\ &= \left| \sum_{j \in [n!]} LCS(\pi_{i_0}, \pi_j) r_j \right| \\ &\leq \sum_{j \in [n!]} LCS(\pi_{i_0}, \pi_j) \end{aligned}$$

$$\begin{aligned}
&= \sum_{j \in [n!]} LIS(\pi_{i_0}^{-1} \pi_j) \\
&= \sum_{\pi \in S_n} LIS(\pi),
\end{aligned}$$

with equality if and only if all the r_j 's have the same sign and have absolute value equal to 1. \square

This gives a trivial bound on the smallest negative value $\lambda_1^{(n)}$: namely, $\lambda_1^{(n)} \geq -\sum_{\pi \in S_n} LIS(\pi)$. Moreover, since the expectation of the longest increasing subsequence of a uniform random permutation is asymptotically $2\sqrt{n}$, this gives an asymptotic order of $-n!\sqrt{n}$ for the lower bound. On the other hand, we are interested in an upper bound for $\lambda_1^{(n)}$. The next result shows that $\lambda_1^{(n)}$ decreases at least exponentially fast.

Proposition 3.2. $\lambda_1^{(n)} \leq 2^{n-4} \lambda_1^{(4)} = -2^{n-3} < 0$.

Proof. This is proved by showing that $\lambda_1^{(n+1)} \leq 2\lambda_1^{(n)}$. As well known,

$$\lambda_1^{(n+1)} = \min_{E^T R = 0} \frac{R^T L^{(n+1)} R}{R^T R}. \quad (3.1)$$

Let $\lambda_1^{(n)}$ be the smallest eigenvalues of $L^{(n)}$ and let $R^{(n)}$ be the corresponding eigenvector. Then, in generating $L^{(n+1)}$ from $L^{(n)}$ as done in the proof of Lemma 2.1, the $n! \times n!$ principal minor of $L^{(n+1)}$ is $L^{(n)} + EE^T$, while its bottom-left $n! \times n!$ submatrix is $L^{(n)}$. Symmetrically, it can be proved that the top-right $n! \times n!$ submatrix is also $L^{(n)}$, while the bottom-right $n! \times n!$ submatrix is $L^{(n)} + EE^T$, i.e., $L^{(n+1)}$ is

$$\begin{bmatrix} L^{(n)} + EE^T & \cdots & L^{(n)} \\ \vdots & \ddots & \vdots \\ L^{(n)} & \cdots & L^{(n)} + EE^T \end{bmatrix}.$$

Further, let

$$R = \begin{bmatrix} R_1^{(n)} \\ 0 \\ \vdots \\ 0 \\ R_1^{(n)} \end{bmatrix}.$$

Then $E^T R = E^T R_1^{(n)} + E^T R_1^{(n)} = 0$, where, by an abuse of notation, E denotes the vector only made up of ones and of the proper dimension. Also,

$$\begin{aligned}\|R\|^2 &= R^T R \\ &= 2 \left\| R_1^{(n)} \right\|^2 \\ &= 2.\end{aligned}$$

In (3.1), the corresponding numerator $R^T L^{(n+1)} R$ is

$$\begin{aligned}\begin{bmatrix} R_1^{(n)} \\ 0 \\ \vdots \\ 0 \\ R_1^{(n)} \end{bmatrix}^T \begin{bmatrix} L^{(n)} + EE^T & \dots & L^{(n)} \\ \vdots & \ddots & \vdots \\ L^{(n)} & \dots & L^{(n)} + EE^T \end{bmatrix} \begin{bmatrix} R_1^{(n)} \\ 0 \\ \vdots \\ 0 \\ R_1^{(n)} \end{bmatrix} \\ = 2 \left(R_1^{(n)} \right)^T (L^{(n)} + EE^T) \left(R_1^{(n)} \right) + 2 \left(R_1^{(n)} \right)^T L^{(n)} \left(R_1^{(n)} \right) \\ = 4 \left(R_1^{(n)} \right)^T L^{(n)} \left(R_1^{(n)} \right) = 4\lambda_1^{(n)}.\end{aligned}$$

Thus,

$$\lambda_1^{(n+1)} \leq 2\lambda_1^{(n)}.$$

□

By a very similar scheme, it can also be proved that the second largest eigenvalue $\lambda_{n!-1}^{(n)}$, which is positive, grows at least exponentially fast.

Proposition 3.3. $\lambda_{n!-1}^{(n)} \geq 2^{n-2} \lambda_1^{(2)} = 2^{n-2} > 0$.

Proof. Using the identity

$$\lambda_{(n+1)!-1}^{(n+1)} = \max_{E^T R=0} \frac{R^T L^{(n+1)} R}{R^T R},$$

with a particular choice of

$$R = \begin{bmatrix} R_{n!-1}^{(n)} \\ 0 \\ \vdots \\ 0 \\ R_{n!-1}^{(n)} \end{bmatrix},$$

where $R_{n!-1}^{(n)}$ is the eigenvector associated with the second largest eigenvalue $\lambda_{n!-1}^{(n)}$ of $L^{(n)}$, leads to $\lambda_{(n+1)!-1}^{(n+1)} \geq 2\lambda_{n!-1}^{(n)}$ and thus proves the result. \square

The above bounds for $\lambda_1^{(n)}$ and $\lambda_{n!-1}^{(n)}$ are far from tight even as far as their asymptotic orders are concerned. Numerical evidence is collected in the following table:

n	$\lambda_1^{(n)}$	$\lambda_1^{(n+1)}/\lambda_1^{(n)}$	$\lambda_{n!-1}^{(n)}$	$\lambda_{(n+1)!-1}^{(n+1)}/\lambda_{n!-1}^{(n)}$
4	-2	1	6.6055	1
5	-5.0835	2.5417	30.0293	4.5460
6	-20.2413	3.9817	166.1372	5.5324
7	-102.9541	5.0860	1083.7641	6.5233

A reasonable conjecture will be that both the smallest and the second largest eigenvalues grow at a factorial-like speed. More precisely, we believe that

$$\lim_{n \rightarrow +\infty} \frac{\lambda_1^{(n+1)}}{\lambda_1^{(n)}(n-1)} = c_1 \geq 1,$$

and that

$$\lim_{n \rightarrow +\infty} \frac{\lambda_{(n+1)!-1}^{(n+1)}}{\lambda_{n!-1}^{(n)}(n+1/2)} = c_2 \geq 1.$$

4 Concluding Remarks

The \sqrt{n} conjecture of Bukh and Zhou is still open and seems quite reasonable in view of the fact that $\mathbb{E}LCS(\sigma_1, \sigma_2) \sim 2\sqrt{n}$, in case σ_1 is uniform and σ_2 arbitrary (again, see [3]). We do not have a proof of this conjecture, but let us nevertheless present a quick $\sqrt[3]{n}$ lower bound result here.

We start with a lemma describing a balanced property among the lengths of *LCSs* of the pairs of any three arbitrary deterministic permutations. This result is essentially due to Beame and Huynh-Ngoc ([1]).

Lemma 4.1. *For any $\pi_i \in \mathcal{S}_n$ ($i = 1, 2, 3$),*

$$LCS(\pi_1, \pi_2)LCS(\pi_2, \pi_3)LCS(\pi_3, \pi_1) \geq n.$$

Proof. The proof of Lemma 5.9 in [1] applies here with slight modification. We further note that this inequality is tight, since letting $\pi_1 = \pi_2 = id$ and $\pi_3 = rev(id)$, which is the reversal of the identity permutation gives, $LCS(\pi_1, \pi_2)LCS(\pi_2, \pi_3)LCS(\pi_3, \pi_1) = n$. \square

In Lemma 4.1, taking $(\pi_1, \pi_2) = (id, rev(id))$ gives, for any third permutation π_3 , $LCS(id, \pi_3)LCS(rev(id), \pi_3) \geq n/LCS(id, rev(id)) = n$. But, since $LCS(id, \pi_3)$ and $LCS(rev(id), \pi_3)$ are respectively the lengths of the longest increasing/decreasing subsequences of π_3 , this lemma can be considered to be a generalization of a well-known result of Erdős and Szekeres (see [4]).

We are now ready for the cubic root lower bound.

Theorem 4.2. *Let P be an arbitrary probability distribution on \mathcal{S}_n and let σ_1 and σ_2 be two i.i.d. random permutations sampled from P . Then $\mathbb{E}_P[LCS(\sigma_1, \sigma_2)] \geq \sqrt[3]{n}$.*

Proof. Let π_1, π_2 and $\pi_3 \in \mathcal{S}_n$ and set $L(\pi_i) := \sum_{\pi_1 \in \mathcal{S}_n} p(\pi_1)LCS(\pi_1, \pi_i) = \sum_{\pi_1 \in \mathcal{S}_n} LCS(\pi_i, \pi_1)p(\pi_1)$, $i = 2, 3$. Then,

$$\begin{aligned} & L(\pi_2) + LCS(\pi_2, \pi_3) + L(\pi_3) \\ &= \sum_{\pi_1 \in \mathcal{S}_n} p(\pi_1)(LCS(\pi_1, \pi_2) + LCS(\pi_2, \pi_3) + LCS(\pi_3, \pi_1)) = 3\sqrt[3]{n} \sum_{\pi_1 \in \mathcal{S}_n} p(\pi_1) = 3\sqrt[3]{n}, \end{aligned} \tag{4.1}$$

by the arithmetic mean-geometric mean inequality and the previous lemma. Further, summing over $p(\pi_2)$ in (4.1) gives:

$$\begin{aligned} & \sum_{\pi_2 \in \mathcal{S}_n} p(\pi_2)(L(\pi_2) + LCS(\pi_2, \pi_3) + L(\pi_3)) \\ &= \sum_{\pi_2 \in \mathcal{S}_n} p(\pi_2)L(\pi_2) + L(\pi_3) + L(\pi_3) \geq 3\sqrt[3]{n}. \end{aligned}$$

Repeating this last procedure but with weights over $p(\pi_3)$ leads to

$$\sum_{\pi_2 \in \mathcal{S}_n} p(\pi_2)L(\pi_2) + 2 \sum_{\pi_3 \in \mathcal{S}_n} p(\pi_3)L(\pi_3) = 3 \sum_{\pi \in \mathcal{S}_n} p(\pi)L(\pi) \geq 3\sqrt[3]{n}. \tag{4.2}$$

However,

$$\mathbb{E}_P[LCS(\sigma_1, \sigma_2)] = \sum_{\pi_1 \in \mathcal{S}_n} \sum_{\pi_2 \in \mathcal{S}_n} p(\pi_1)LCS(\pi_1, \pi_2)p(\pi_2)$$

$$\begin{aligned}
&= \sum_{\pi_1 \in \mathcal{S}_n} p(\pi_1) \sum_{\pi_2 \in \mathcal{S}_n} LCS(\pi_1, \pi_2) p(\pi_2) \\
&= \sum_{\pi \in \mathcal{S}_n} p(\pi) L(\pi).
\end{aligned}$$

Combining this last identity with (4.2) proves the result. \square

The above proof is simple: it basically averages out each $LCS(\cdot, \cdot)$ as $\sqrt[3]{n}$ on the summation weighted by P , but in view of the original conjecture, the cubic root lower bound is not tight. Apart from our curiosity concerning this \sqrt{n} conjecture, it would be interesting to know the exact asymptotic order of the smallest eigenvalue $\lambda_1^{(n)}$ of $L^{(n)}$, in particular when compared with our full knowledge of the largest eigenvalue $\lambda_{n!}^{(n)}$, which is of order as $2n!\sqrt{n}$, by its inherent connection to the length of LISs of a uniform random permutation of $[n]$. Even further, an ultimate goal would be to have a complete knowledge of the distribution of all the eigenvalues of $L^{(n)}$. These problems/questions are natural generalizations of similar problems/questions in the study of the longest increasing subsequences of uniform permutations.

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